ON TURNING OVER GYROSCOPIC PRECESSION

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Abstract

It is said that the vorticity of a congruence plays the role of rate of rotation for the precession of a gyroscope moving along a world-line belonging to the congruence. Our aim is to determine the evolution equation for the angular momentum of a gyrosocope with respect to an arbitrary time-like congruence, i.e, a reference congruence which does not contain the curve described by the gyroscope. In particular, we will show what conditions are needed to support the previous assert about the vorticity. So, we establish a well-founded theoretical description for the analysis of the precession of gyroscopes, providing suitable conclusions for an eventual performance of planned experiments.

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1. INTRODUCTION

The analysis of the evolution experimented by the angular momentum of a gyroscope in the presence of a gravitational field is a problem in General Relativity which has been recovered a full interest induced by the near expected (althoguh delayed) launching of the artificial satelite *Gravity Probe B* (GP-B) [1]. It is so that, in the last years a series of articles [2]-[8], studying different issues of this subject, have been written which show explicitely that, in spite of the time passed from the pioneering works were published [9]-[12], there are still some aspects which are not well-known requiring to shed light on them.

Among those of the published articles, we must stand up for the one by Rindler and Perlick [2]. There, the foundations are established for studying the evolution of a point-like gyroscope (i.e., with neglected dimensions) with respect to a reference system considered as a congruence of time-like world-lines. This point of view is essential in order to bring out either a model of structure supporting the gyroscope or whatever other reference which can be used to evaluate the physical magnitude involved in the problem. Moreover, an specific procedure is proposed in this article, which allows to recover fairly simply the classical post-newtonian results due to Fokker–de Sitter and Schiff.

The aim of the present work is to serve as a complementary analysis respect to the previously mentioned study by Rindler and Perlick [2] into a fundamental aspect: to analyze the evolution of the gyroscope's angular momentum with respect to an arbitrary congruence, that is to say, a congruence which does not contain the curve along which the gyroscope is moving. In this way, it shows how the intrinsic quantities defined by the congruence on the quotient manifold (acceleration or Newtonian field, rotation or Coriolis field and deformation rate) affect the evolution of the gyroscope. In particular, the result of Rindler and Perlick is obtained when the gyroscope is supposed to be at rest with respect to the congruence which, in addition, has not deformation rate (Born's congruence). This result implies, as they assert, that the precession rate of the gyroscope coincides with the rotation tensor of the congruence.

In paper [2] the authors are interested basically in the final rotated angle by the gyrosocope after a period in a closed orbit, without taking into account the influence of the gravitational fields associated to an arbitrary congruence of reference. For this reason, the problem concerning the change from the "proper" congruence to an arbitrary one is intuitively solved by taking on level the standard coordinates of Schwarzschild metric (and the Boyer–Lindquist coordinates of the Kerr metric) with the spherical coordinates of the Minkowski flat space. In this way, it is assumed that the tangent vectors to the radial coordinates rotate an angle 2π when a displacement of one revolution is carried along a circular orbit in the equatorial plane. We supply a rigorous proposal consisting of defining previously a new angular momentum by means of a suitable boost. Next, we evaluate the rotation of this spin (new angular momentum) with respect to a triad which is transported by paralellism in the sense of the quotient metric associated to the congruence. Finally, it is possible to recover the results obtained by Rindler and Perlick if the proper rotation of the triad is considered for a complete revolution of the circular orbit.

The whole of the work will be presented as follows. Section 2 is devoted to the study of the time-like congruence of space-time, emphasizing those aspects that will be used later, as in particular, the introduction of the concept of natural orthogonal connectors triad.

The section 3 is divided into three subsections. The first one is devoted to the evolution of a gyroscope with respect to a congruence and the construction of a new spin by means of a local Lorentz's boost. In the second subsection the evolution equation of the new spin in the three-dimensional formalism associated to the reference congruence is obtained. Finally, in the third subsection a field of non-natural connectors triad is introduced in order to generalize the cartesian connectors of the Minkowski space-time.

Finally, in order to illustrate the matter we evaluate in Section 4 the precession for Schwarzschild and Kerr metrics, where our aim is not only to show that this methodology provides the right well-known results, but at the same time to notice the difficulties that would be induced by other managements of this problem in more general cases.

2. TIME-LIKE CONGRUENCES

In this section we are going to introduce explicitely the definitions that we will use further concerning with time–like congruence. Let $\mathcal{C}$ be a congruence of time-like world-lines on some domain $\mathcal{D}$ of the space-time manifold $(\mathcal{V}_4\,, g_{\alpha\beta})$, and let

$$x^\alpha = f^\alpha(p\,, z^i) \tag{1}$$

$(i, j, ... = 1, 2, 3;\ \alpha, \beta, ... = 0, 1, 2, 3)$ be the parametric equations of this congruence. The unit time-like tangent vector field is

$$u^\alpha(x) = \xi^{-1}\,\xi^\alpha[p(x), z^i(x)] \quad , \quad g_{\alpha\beta}u^\alpha u^\beta = -1 \tag{2}$$

with

$$\xi^\alpha \equiv \frac{\partial f^\alpha}{\partial p} \quad , \quad g_{\alpha\beta}\xi^\alpha\xi^\beta \equiv -\xi^2 < 0 \tag{3}$$

and where

$$\begin{cases} p = p(x^\alpha) \\ z^i = z^i(x^\alpha) \end{cases} \tag{4}$$

are the inverse functions of (1). The time-like parameter p of the congruence can be choosen, up to the gauge transformation $\tau_{_R} \to \tau_{_R} + A(z^i)$, to be the proper time of the congruence such that

$$\tau_{_R} : \; u^\alpha = \frac{\partial x^\alpha}{\partial \tau_{_R}} \tag{5}$$

The three tensor fields associated intrinsically with the time-like congruence, Deformation rate $\Sigma_{\alpha\beta}$, Rotation $\Omega_{\alpha\beta}$ (so called sometimes Coriolis or Gravitomagnetic field) and Acceleration b_α (Newtonian or Gravitoelectric field up to sign), are given by

$$\nabla_\alpha u_\beta = \Sigma_{\alpha\beta} + \Omega_{\alpha\beta} - u_\alpha b_\beta \tag{6}$$

$$\begin{cases} \Sigma_{\alpha\beta} = \dfrac{1}{2}\hat{g}^\lambda_\alpha\,\hat{g}^\mu_\beta(\nabla_\lambda u_\mu + \nabla_\mu u_\lambda) \\ \Omega_{\alpha\beta} = \dfrac{1}{2}\hat{g}^\lambda_\alpha\,\hat{g}^\mu_\beta(\nabla_\lambda u_\mu - \nabla_\mu u_\lambda) \\ b_\alpha = u^\rho \nabla_\rho u_\alpha \end{cases} \tag{7}$$

where

$$\hat{g}^\lambda_\alpha \equiv \delta^\lambda_\alpha + u^\lambda u_\alpha \tag{8}$$

is the proyector tensor orthogonal to u^α (usually considered the metric on the 3-space quotient manifold).

As we will be concerned with the evolution of a vector orthogonal to u^α, we firstly define a 3–frame orthogonal to u^α to refer it. The natural way to do that is as follows: since each line of the congruence is characterized by a parameter z^i, a vector connecting two lines of (1) and orthogonal to u^α is given by a linear combination (with functions independent of the parameter p) of the spatial projection of the three derivatives

$$Q^\alpha_i \equiv \frac{\partial f^\alpha}{\partial z^i} \tag{9}$$

i.e.

$$q_i^\alpha \equiv \hat{g}_\lambda^\alpha \, Q_i^\lambda = \hat{g}_\lambda^\alpha \, \frac{\partial f^\lambda}{\partial z^i} \tag{10}$$

which, from here on, we will call the *natural orthogonal connectors triad* $\{q_i^\alpha\}$. They are the components of the following vectors-field

$$\hat{\partial}_i \equiv q_i^\alpha \frac{\partial}{\partial x^\alpha} = \varphi_i \frac{\partial}{\partial p} + \frac{\partial}{\partial z^i} \quad , \quad \varphi_i \equiv \xi^{-1}(u \, Q_i) \tag{11}$$

So, a tetrad of space-time is given by $\{e_a^\alpha\} \equiv \{u^\alpha, q_i^\alpha\}$, $(a, b, ... = 0, 1, 2, 3)$, which satisfies the orthogonality conditions

$$g_{\alpha\beta} \, u^\alpha \, q_i^\beta = 0 \tag{12}$$

and where

$$g_{\alpha\beta} \, q_i^\alpha \, q_j^\beta \equiv \hat{g}_{ij} \tag{13}$$

are the components of the quotient metric with respect to the triad $\{q_i^\alpha\}$. The related co-base of one-forms can be constructed to give $\{\theta_\alpha^a\} \equiv \{-u_\alpha \, , \, p_\alpha^i\}$, such that

$$\begin{cases} p_\alpha^i dx^\alpha = \dfrac{\partial z^i}{\partial x^\alpha} dx^\alpha = dz^i \\ \\ q_i^\alpha \, p_\alpha^j = \delta_j^i \end{cases} \tag{14}$$

It is worthwhile to note the behaviour of the orthogonal connectors triad under a change of parameters of the congruence (1). If a general change is performed, like the following

$$\begin{cases} p \rightarrow p' = p'(p, z^i) \\ \\ z^i \rightarrow z^{k'} = z^{k'}(z^i) \end{cases} \tag{15}$$

then we have

$$Q_i^\alpha = \xi' u^\alpha \frac{\partial p'}{\partial z^i} + Q_{k'}^\alpha \frac{\partial z^{k'}}{\partial z^i} \tag{16}$$

with

$$\xi' = \left(\frac{\partial p'}{\partial p} \right)^{-1} \xi \tag{17}$$

and therefore, the natural orthogonal connector changes like a tensor on the quotient manifold, no matter the time parameter p' considered,

$$q_i^\lambda = q_{k'}^\lambda \frac{\partial z^{k'}}{\partial z^i} \tag{18}$$

It will be usefull for further calculations to obtain the coefficients γ^c_{ab} of the connection, with respect to the tetrad $\{e^\alpha_a\} \equiv \{u^\alpha, q^\alpha_i\}$, in terms of the geometrical objects defining the congruence. As it is well known we have

$$e^\lambda_b \nabla_\lambda e^\mu_a = \gamma^c_{ab}\, e^\mu_c \tag{19}$$

and, on the other hand, we also have the following

$$d\theta^c = -\frac{1}{2} C^c_{ae} \theta^a \wedge \theta^e \tag{20}$$

C^c_{ae} being the coefficients of the Lie's algebra for the tetrad vectors, that is to say

$$[\vec{e}_a, \vec{e}_b] = C^c_{ab} \vec{e}_c \tag{21}$$

Using now the definition of connectors we obtain

$$\begin{cases} [\vec{u}\,, \vec{q}_j] = (b\, q_j)\, \vec{u} \\ \\ [\vec{q}_i\,, \vec{q}_j] = 2\, \hat{\Omega}_{ij}\, \vec{u} \end{cases} \tag{22}$$

and then it is easy to conclude in the following expressions for the coefficientes of the connection

$$\begin{cases} \gamma^0_{00} = 0 \quad , \quad \gamma^0_{0k} = 0 \quad , \quad \gamma^0_{j0} = (b\, q_j) \quad , \quad \gamma^0_{jk} = \hat{\Sigma}_{kj} + \hat{\Omega}_{kj} \\ \\ \gamma^i_{00} = \hat{b}^i \quad , \quad \gamma^i_{j0} = \gamma^i_{0j} = \hat{\Sigma}_j{}^i + \hat{\Omega}_j{}^i \quad , \quad \gamma^i_{jk} = \gamma^i_{kj} \equiv \hat{\Gamma}^i_{jk} \end{cases} \tag{23}$$

where the following notation has been used (the latin indexes are raised and lowered with the quotient metric $\hat{g}_{ij}$)

$$\begin{cases} \hat{b}_i \equiv q^\alpha_i\, b_\alpha \\ \\ \hat{\Omega}_{ij} \equiv q^\alpha_i\, q^\beta_j\, \Omega_{\alpha\beta} = \dfrac{1}{2}\xi(\hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i) \\ \\ \hat{\Sigma}_{ij} \equiv q^\alpha_i\, q^\beta_j\, \Sigma_{\alpha\beta} = \dfrac{1}{2}\xi^{-1} \dfrac{\partial \hat{g}_{ij}}{\partial p} \end{cases} \tag{24}$$

and where $\hat{\Gamma}^i_{jk}$ is the Zel'manov-Cattaneo connection [13]-[15]

$$\hat{\Gamma}^i_{jk} = \frac{1}{2} \hat{g}^{ih} (\hat{\partial}_j \hat{g}_{kh} + \hat{\partial}_k \hat{g}_{jh} - \hat{\partial}_h \hat{g}_{jk}) \tag{25}$$

3. GYROSCOPE'S EVOLUTION WITH RESPECT TO AN ARBITRARY CONGRUENCE

Let us now consider a point-like gyroscope (neglectable size) moving along an arbitrary time-like curve. Hence, as usual, we can clear away the Papapetrou equations [16] and suppose that the spin of the gyroscope is Fermi-Walker transported along the curve. The aim of this section is to describe exactly the evolution of this spin with respect to a certain reference time-like congruence, and so in terms of the geometric objects associated to it: acceleration, rotation, deformation rate and Cattaneo's connection. This generalize, for example, the conclusions concerning the postnewtonian experiment of a rocket–gyroscope orbit around the Earth [17]. In adittion, as we will see, it is not a trivial question to recover the well known Thomas or Fokker–de Sitter and Schiff postnewtonian precession terms, because the evaluation of the rotated angle by the spin of the gyroscope is not a matter of superficial geometric considerations.

A) Fermi-Walker transport and local covariant Lorentz's boost

Let S^α be a Fermi-Walker transported (FWT) vector along a time-like curve with unit tangent vector w^α and acceleration a^α:

$$x^\alpha = \varphi^\alpha(\tau) : \quad \begin{cases} w^\alpha \equiv \dfrac{d\varphi^\alpha}{d\tau} \\[2ex] a^\alpha \equiv \dfrac{\nabla w^\alpha}{d\tau} \end{cases} \quad , \quad g_{\alpha\beta} w^\alpha w^\beta = -1 \tag{26a}$$

$$\frac{\nabla S^\alpha}{d\tau} + (a^\alpha\, w_\lambda - a_\lambda\, w^\alpha)\, S^\lambda = 0 \tag{26b}$$

Two first integrals of this differential equation are given by $S_\mu w^\mu$ and the length of S^α, i.e. $g_{\alpha\beta} S^\alpha S^\beta$. Therefore we can take $S_\mu w^\mu = 0$, and so, from here on S^α represents the intrinsic spin vector of a gyroscope moving along such a curve.

Let $\mathcal{C}$ be now a congruence of time-like word-lines defined as usually

$$\mathcal{C} : \; x^\alpha = f^\alpha(\tau_{_R}, z^i) \quad , \quad \begin{cases} \tau_{_R} = \tau_{_R}(x^\alpha) \\[1ex] z^i = z^i(x^\alpha) \end{cases} \tag{27}$$

being $\tau_{_R}$ the proper time and u^α the unit tangent vetor

$$u^\alpha(x) = \frac{\partial f^\alpha}{\partial \tau_{_R}}[\tau_{_R}(x), z^i(x)] \quad , \quad g_{\alpha\beta} u^\alpha u^\beta = -1 \tag{28}$$

An observer evolving with the reference congruence see the gyroscope moving because in general u^α is not co-linear with w^α. Then, one needs to know what are the components of the spin vector S^α as seen by this observer. To do that we define a covariant local Lorentz's boost $\mathcal{B} : \; \{u^\alpha\} \longrightarrow \{w^\alpha\}$, transforming this observer at rest for the congruence, into the observer tied to the gyroscope. By applying the boost to the spin-vector S^α we get

$$S^\alpha \longrightarrow N^\alpha = S^\alpha + \frac{(Su)+(Sw)}{1-(uw)}(u^\alpha + w^\alpha) - 2(Sw)u^\alpha \tag{29}$$

This transformation fulfills the following properties

$$\begin{cases} (Nu) = (Sw) \\ \quad N^2 = S^2 \\ (Nw) = -(Su) - 2(Sw)(uw) \end{cases} \tag{30}$$

In particular, as it is shown, it preserves the length of the spin-vector. Moreover, because of the spin-vector S^α is orthogonal to w^α we have

$$\begin{cases} (Nu) = 0 \\ (Nw) = -(Su) \end{cases} \tag{31}$$

and so the new spin vector N^α is orthogonal to the congruence of reference.

Let us now write down the evolution equation of the transformed spin-vector N^α along de curve $\varphi^\alpha(\tau)$. Since S^α verify the FWT equation (26b) it is straightforward to obtain from (29) that

$$\begin{aligned} \frac{\nabla N^\alpha}{d\tau} =& \frac{(Na)}{1+\gamma}(w^\alpha - \gamma u^\alpha) - \frac{(Nw)}{1+\gamma}\left[a^\alpha + (ua)u^\alpha\right] \\ &+ \frac{1}{1+\gamma}\left[N_\rho(u^\alpha + w^\alpha) - (Nw)\delta^\alpha_\rho\right]\frac{\nabla u^\rho}{d\tau} \end{aligned} \tag{32}$$

with

$$\frac{\nabla u^\rho}{d\tau} = w^\lambda({\Sigma_\lambda}^\alpha + {\Omega_\lambda}^\alpha - u_\lambda b^\alpha) \tag{33}$$

and where $\gamma \equiv -(u\ w)$ is analogous to the factor appearing in the Lorentz's transformation for the Special Relativity.

B) Evolution equation of the spin N^α in three dimensional formalism

We will refer now N^α, as well as the tangent unit vector w^α and the acceleration a^α of the curve, to the tetrad of space–time $\{u^\alpha, q_i^\alpha\}$

$$\begin{cases} N^\alpha = \hat{N}^i\, q_i^\alpha \\ w^\alpha = \gamma\, u^\alpha + \hat{w}^i\, q_i^\alpha \\ a^\alpha = -(ua)u^\alpha + \hat{a}^i\, q_i^\alpha \end{cases} \tag{34}$$

From the first expression of (34) we can write down the covariant derivative of N^α in this way

$$\frac{\nabla N^\alpha}{d\tau} = \frac{d\hat{N}^i}{d\tau}\, q_i^\alpha + \hat{N}^i\, \frac{\nabla q_i^\alpha}{d\tau} \tag{35}$$

where $\dfrac{\nabla q_i^\alpha}{d\tau}$, using the second expression of the decomposition (34), is

$$\frac{\nabla q_i^\alpha}{d\tau} = w^\rho \nabla_\rho q_i^\alpha = \gamma\, u^\rho \nabla_\rho q_i^\alpha + \hat{w}^k q_k^\rho\, \nabla_\rho q_i^\alpha \tag{36}$$

By using now the coefficients (24) of the connection (Ricci's rotation coefficients) with respect to the tetrad above we have

$$\begin{aligned} \frac{\nabla q_i^\alpha}{d\tau} = &- (uw)\left[(bq_i)u^\alpha + q_i^\lambda(\Sigma_\lambda{}^\alpha + \Omega_\lambda{}^\alpha)\right] \\ &+ \hat{w}^k\left[(\hat{\Sigma}_{ki} + \hat{\Omega}_{ki})u^\alpha + \hat{\Gamma}^j_{ki} q_j^\alpha\right] \end{aligned} \tag{37}$$

With this result and (32) we can extract from (35) the evolution equation of the components of N^α in the connectors frame $\{q_i^\alpha\}$:

$$\frac{d\hat{N}^i}{d\tau} + \hat{\Gamma}^i_{kj}\hat{w}^j\hat{N}^k = B^i_k\hat{N}^k \tag{38a}$$

being,

$$\begin{aligned} B^i_k \equiv\, &\gamma(\hat{\Omega}^i{}_k - \hat{\Sigma}^i{}_k) \\ &+ \frac{1}{1+\gamma}\left[\hat{w}^i(\gamma\hat{b}_k + \hat{a}_k) - \hat{w}_k(\gamma\hat{b}^i + \hat{a}^i)\right] \\ &+ \frac{1}{1+\gamma}\left[\hat{w}^i\hat{w}^j(\hat{\Sigma}_{jk} + \hat{\Omega}_{jk}) - \hat{w}_k\hat{w}^j(\hat{\Sigma}_j{}^i + \hat{\Omega}_j{}^i)\right] \end{aligned} \tag{38b}$$

The equation (38) describes the evolution of a gyroscope moving along a curve with 3-velocity $\hat{w}^i$ with respect to an arbitrary congruence. As can be seen, the expression involves not only geometric objects of the congruence (as rotation, deformation, acceleration or Zel'manov–Cattaneo connection) but also the velocity and acceleration of the curve described by the gyroscope. On the other hand the equation (38) is not in general a "precession" equation, but something more complicated, because of the presence of $\hat{\Sigma}_{ij}$ and the symmetric part of the covariant component of $\hat{\Gamma}^i_{kj}\hat{w}^j$. In this sense, it is suitable for further aplications to write down the equation (38) in their separated symmetric and antisymmetric parts, i.e,

$$\frac{d\hat{N}^i}{d\tau} = (\mathcal{A}^i_k + \mathcal{S}^i_k)\hat{N}^k \tag{39a}$$

with

$$\mathcal{A}_{jk} \equiv \hat{g}_{ji}\mathcal{A}^i_k \equiv \gamma\,\hat{\Omega}_{jk} + \frac{1}{1+\gamma}\left[\hat{w}_j(\gamma\hat{b}_k + \hat{a}_k) - \hat{w}_k(\gamma\hat{b}_j + \hat{a}_j)\right]$$

$$+ \frac{1}{1+\gamma}\left[\hat{w}_j(\hat{\Sigma}_{lk} + \hat{\Omega}_{lk}) - \hat{w}_k(\hat{\Sigma}_{lj} + \hat{\Omega}_{lj})\right]\hat{w}^l \tag{39b}$$

$$+ \frac{1}{2}(\hat{\partial}_j\hat{g}_{kl} - \hat{\partial}_k\hat{g}_{jl})w^l$$

$$\mathcal{S}_{jk} \equiv \hat{g}_{ji}\mathcal{S}^i_k \equiv -\gamma\,\hat{\Sigma}_{jk} - \frac{1}{2}\hat{w}^l\hat{\partial}_l\hat{g}_{jk} \tag{39c}$$

The expression (39b) shows a precession rate which generalizes the classical rate of precession appearing into the postnewtonian calculations (see for instance Weinberg [18]), since the geodetic-precession terms (gravitational Thomas's precession) and those refering the spin-spin interaction (gravitomagnetic effects) are obviously more complicated.

Let us note that the equation (38) is a tensorial expression on the quotient manifold since the left hand side is the covariant derivative of $\hat{N}^i$ in the sense of Zel'manov–Cattaneo connection. However, the evaluation of the rotated angle by the vector $\hat{N}^i$ (see Appendix), after a revolution on a closed orbit, depends on the parametrization of the congruence, because it is defined up to a integer multiple of 2π. As an example, we mention the more simple case of Minkowski space-time. If we consider a gyroscope moving on a circular orbit at the "equatorial" plane and we refer the evolution to the congruence defined by the time coordinate varying alone, which is irrotational and have no deformation rate, equation (38) appears easier. By solving this equation in cartesian coordinates ($\hat{\Gamma}^i_{jk} = 0$) the gyroscope precess in an orbital period an angle $2\pi(\gamma - 1)$, which is the correct Thomas's precession, whereas solving the same equation in spherical coordinates the angle is $2\pi\gamma$. It is clear for the Minkowski space–time why the difference betweeen both calculations of the rotated results to be 2π, since the cartesian connectors evolves by paralelism, whereas the connectors in spherical coordinates rotate exactly 2π after a period of the orbit.

The problem is a little more complicated if we consider, for example, the Schwarzschild or Kerr space–time (also with an equatorial circular orbit and the congruence defined by t varying alone) because the quotient manifold is a non-flat 3–Riemannian metric, and therefore we have not "cartesian connectors" which can be used as a reference.

C) Triads of non natural orthogonal connectors

In order to avoid the ambiguity in the determination of the rotation of the gyroscope in a general case, it is necessary to introduce a "cartesian-like connectors" system on the quotient space. Firstly we define a general triad of connectors as a combination of the orthogonal connectors

$$h_i^\alpha = \hat{h}_i^k q_k^\alpha \quad , \quad \det(\hat{h}_i^k) > 0 \tag{40}$$

where the coefficientes $\hat{h}_i^k$ only depend on the space parameters of the congruence, i.e., $\hat{h}_i^k = \hat{h}_i^k(z^j)$, in order to preserve the meaning given to them. If we now look for changes in the evolution equation of N^α (38), we end up in the following

$$\frac{d\tilde{N}^i}{d\tau} + \tilde{q}_j^i \frac{\hat{\nabla}\hat{h}_k^j}{d\tau} = \tilde{B}_k^i \tilde{N}^k \tag{41}$$

where $\hat{\nabla}$ stands for the covariant derivative with respect to $\hat{\Gamma}^i_{jk}$ and $\tilde{B}_k^i$ denotes the analogue expression to (38b) respect to the new connectors, the "tilde" being a notation for a similar decomposition to (34) but relative to that new connectors h_i^α. The matrix $\tilde{q}_j^i$ denotes the inverse of (40), i. e. $q_k^\alpha = \tilde{q}_k^i h_i^\alpha$.

Obviously the equation (41) becomes simplified by assuming the new connectors to be transported by paralelism, in the sense of $\hat{\Gamma}$, along the curve described by the gyroscope. This assumption implies an inambiguous criterion for the evaluation of the gyroscope–orientation change, although, as we will see, it is necessary to evaluate for each case the holonomy group on the quotient manifold. At the same time it represents a way to generalize intrinsically the connectors associated to cartesian coordinates in the Minkowski space–time.

With this election for the new connectors and putting our attention on Born's congruences ($\tilde{\Sigma}_{ij} = 0$) the equation (41) results to be

$$\frac{d\tilde{N}^i}{d\tau} = \tilde{\mathcal{A}}_k^i \tilde{N}^k \tag{42a}$$

$$\begin{aligned}\tilde{\mathcal{A}}_k^i \equiv \gamma \tilde{\Omega}^i{}_k &+ \frac{1}{1+\gamma}\left[\tilde{w}^i \tilde{w}^j \tilde{\Omega}_{jk} - \tilde{w}_k \tilde{w}^j \tilde{\Omega}_j{}^i\right] \\ &+ \frac{1}{1+\gamma}\left[\tilde{w}^i(\gamma \tilde{b}_k + \tilde{a}_k) - \tilde{w}_k(\gamma \tilde{b}^i + \tilde{a}^i)\right]\end{aligned} \tag{42b}$$

which is an authentic precession equation for the evolution of the gyroscope. We will illustrate this situation in the next Section with the Schwarzschild and the Kerr metrics.

4. GYROSCOPE ORBITING IN SCHWARZSCHILD AND KERR SPACE–TIME

A) Schwarzschild space-time

Let us now the Schwarzschild space-time as an example to calculate the precession of a gyroscope moving in a equatorial circular orbit. In standard coordinates this metric is writen

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{43}$$

On one hand we have the standard reference's time-like congruence

$$\mathcal{C} : \begin{cases} t = p \\ r = z^1 \quad , \quad \theta = z^2 \quad , \quad \phi = z^3 \end{cases} \tag{44}$$

which have associated the following quantities

$$\begin{cases} \xi^\lambda = \dfrac{\partial x^\alpha}{\partial p} = (1\,,0\,,0\,,0) \\ Q_i^\lambda = \dfrac{\partial x^\alpha}{\partial z^i} = \delta_i^\alpha \end{cases} \tag{45}$$

$$\xi^2 = 1 - \frac{2m}{r} \quad , \quad \varphi_i = 0 \tag{46}$$

and

$$\begin{cases} \hat{g}_{ij} = \mathrm{diag}\left(\dfrac{1}{1 - 2m/r}\,, r^2\,, r^2 \sin^2\theta\right) \\ \hat{\Omega}_{ij} = 0 \quad , \quad \hat{\Sigma}_{ij} = 0 \quad , \quad b_i = \dfrac{m/r^2}{1 - \dfrac{2m}{r}}(1\,,0\,,0) \end{cases} \tag{47}$$

On the other hand we consider a circular orbit in the equatorial plane with constant angular velocity

$$\begin{cases} t = p(\tau) \\ r = R \quad , \quad \theta = \dfrac{\pi}{2} \\ \varphi = \omega_\pm \, t \quad , \quad (\omega_\pm = \pm\omega \;,\; \omega = \mathrm{Cte} > 0) \end{cases} \tag{48}$$

where $\pm$ denotes the sense of rotation (direct or retrograde) of the orbit and $p(\tau)$ must be choosen such that $w^\alpha w_\alpha = -1$, i. e.

$$w^\alpha = \frac{dx^\alpha}{d\tau} = \frac{1}{X}(1\,,0\,,0\,,\omega) \tag{49}$$

where

$$X \equiv \left(\frac{dp}{d\tau}\right)^{-1} = \left(1 - \frac{2m}{R} - \omega^2\,R^2\right)^{1/2} \tag{50}$$

So the acceleration of the gyroscope along the orbit turns out to be

$$a^\alpha = \frac{\nabla w^\alpha}{d\tau} = \frac{\xi^2}{X^2}\left(\frac{m}{R^2} - \omega^2 R\right)(0\,,1\,,0\,,0) \tag{51}$$

Since (42) is a tensorial expression with respect to changes of connectors triads type like (40), and as we are only interested in evaluate the precession angle of the gyroscope, we can carry out the calculation by using the connectors associated to the standard parametrization (44). So, we have first of all:

$$\mathcal{A}_{ik} = \frac{1}{1+\gamma}\left[\hat{w}_i(\gamma\hat{b}_k + \hat{a}_k) - \hat{w}_k(\gamma\hat{b}_i + \hat{a}_i)\right] \tag{52}$$

where

$$\gamma = -g_{\alpha\beta}u^\alpha w^\beta = \frac{\xi}{X} \tag{53}$$

and therefore, from (47), (49) and (51)

$$\begin{cases} \mathcal{A}_{12} = \mathcal{A}_{23} = 0 \\ \mathcal{A}_{31} = \mp\dfrac{\omega R}{\xi\,X^2}\left[1 - \dfrac{3m}{R} - \xi\,X\right] \end{cases} \tag{54}$$

The precession angle and the sense of rotation can be obtained from the following dual vector (see Appendix))

$$\Omega^i = -\frac{1}{2}\frac{1}{\sqrt{\hat{g}}}\epsilon^{ijk}\mathcal{A}_{jk} = -\frac{1}{\sqrt{\hat{g}}}\mathcal{A}_{31}\,\delta^i_2 \tag{55}$$

which turns out to be

$$\Omega^i = \pm\frac{\omega}{R\,X^2}\left[1 - \frac{3m}{R} - \xi\,X\right](0\,,1\,,0) \tag{56}$$

- By taking in (56) the orbit to be geodesic ($\omega^2 = m/R^3$), we have

$$\Omega^\theta = \mp \frac{\sqrt{m/R}}{R^2\, X_g} \left[\sqrt{1-\frac{2m}{R}} - \sqrt{1-\frac{3m}{R}} \right] \left\{ \begin{array}{l} <0 \\ >0 \end{array} \right. \tag{57}$$

where X_g stands for the value of X when the orbit is geodesic ($X_g \equiv \sqrt{1-3m/R}$). Hence, the precession is *direct* or *retrograde* for *direct* or *retrograde* orbits respectively, and the rotated angle after a period of proper time turns out to be

$$\Delta\alpha = \pm\Omega\, \frac{2\pi\, X_g}{\omega} = \pm 2\pi \left[\sqrt{1-\frac{2m}{R}} - \sqrt{1-\frac{3m}{R}} \right] \tag{58}$$

where the corresponding sign is considered for the respective sense of the orbit.

• On other hand, by taking $m = 0$ in the expression (56) we can recover the result for Minkowski space-time. As it is known, for the case of a *direct* orbit the precession is *retrograde* and reciprocaly, which is clear from the component

$$\Omega^\theta = \frac{\pm\omega}{R\,(1-\omega^2\, R^2)} \left[1 - \sqrt{1-\omega^2\, R^2} \right] \left\{ \begin{array}{l} >0 \\ <0 \end{array} \right. \tag{59}$$

and the rotated angle is

$$\Delta\alpha \equiv \mp\Omega\, \frac{2\pi\, X_{_m}}{\omega} = \mp 2\pi \left[(1-\omega^2\, R^2)^{-1/2} - 1 \right] \tag{60}$$

being $X_{_m}$ the value of X (50) for $m = 0$, and where the signs $\mp$ are for *direct* or *retrograde* orbits respectively.

As can be checked, results (58) and (60) do not correspond with the expressions obtained by Rindler and Perlick [2]. The expressions obtained by these authors simply come from adding an angle 2π to the result obtained if it is considered the congruence defined by all the circular orbits centered in the symmetry axies and "orthogonal" to it (in this way the matrix B^i_k of (38) is reduced to $\hat\Omega^i{}_k$ and $\hat\Gamma^i_{kj} = 0$). This procedure is quite reasonable for simple examples, but it does not seem to be supported in the general case, and moreover it only gives the rotated angle but it allows to conclude nothing about the influence of the associated fields to the reference congruence. The question anyway is the following: what a criterion is used to say that the angle of precession is anyone?. A possible answer for this question comes from the use of a triad of connectors transported by paralelism in the sense of $\hat g_{ij}$, as we have showed, but obviously there is not an unique answer and some other likewise reasonable criterion may be used.

With respect to the difference between results (58), (60) and those obtained by Rindler and Perlick, it is worthwhile to note that it is due to the fact that the connectors transported by paralellism along a closed curve in a Riemannian space rotate certain angle (holonomy), which is zero in a flat Minkwosky space-time. Therefore, we rely on a procedure to calculate the "total" angle that the gyrosocpe rotates by adding the rotated angle by the paralell transported connectors to the expressions (58) and (60). For the evaluation of the rotated angle, after a revolution, by the triad transported by paralellism we avoid the possible ambiguity by forcing the angle to be zero in the Minkowkian limit.

In order to do that, let us consider the paralell transport for the connectors in the quotient space of Schwarzschild space-time. The final equation is

$$\frac{dq^i}{d\tau} = P^i{}_j q^j \tag{61}$$

with

$$P^i{}_j \; : \quad P^i \equiv -\frac{1}{2}\frac{1}{\sqrt{\hat{g}}}\epsilon^{ijk}\hat{g}_{jl}P^l{}_k = \pm\frac{\omega}{R}\frac{\sqrt{1-\frac{2m}{R}}}{\sqrt{1-\frac{3m}{R}}}\,(0,1,0) \tag{62}$$

By using the thechnics appearing in the Appendix it is trivial to conclude that the connector rotates, after a period T_p, an angle

$$|\Delta\beta| \equiv \frac{2\pi}{T_p}\left(P^i P_i\right)^{1/2} = \left|2\pi\sqrt{1-\frac{2m}{R}} + 2k\pi\right| \tag{63}$$

Since this angle must be zero in the Minkowskian limit, we have that the corresponding angles for the respective *direct* and *retrograde* orbit case result to be

$$\Delta\beta = \pm 2\pi\left[1-\sqrt{1-\frac{2m}{R}}\right] \tag{64}$$

Hencefore, the precession for the geodesic case could be corrected as follows

$$\Delta\alpha + \Delta\beta = \pm 2\pi\left(1-\sqrt{1-\frac{3m}{R}}\right) \tag{65}$$

result that agree with the one obtained by Rindler and Perlick.

B) Kerr space-time

Finally, we would like to complete this analysis by showing the results obtained in the Kerr space-time.

We firstly calculate the precession angle of the gyroscope by using the connectors associated to the standard parametrization (44) of the reference's time-like congruence (which are considered to be transported by paralellism), as well as the circular orbit in the equatorial plane (48), being $\{t, r, \theta, \varphi\}$ Boyer-Lindquist coordinates. In this case, if the orbit is a geodesic, the constant angular velocity which turns out to be

$$\omega_{\pm} = \frac{\omega_s}{a\omega_s \pm 1} \tag{66}$$

with $\omega_s \equiv +\sqrt{m/R^3}$ and where the respective sign denotes the orbit to be *direct* or *retrograde*.

It is straightforward to calculate that (55) becomes:

$$\Omega^\theta = \mp \frac{\omega_s^2}{R\xi^3 X_k} C \tag{67}$$

where

$$\begin{cases} C \equiv -a(a\omega_s \pm \xi^2) \pm \dfrac{\omega_s \xi^2 R^2(\xi^2 + a^2/R^2)}{a\omega_s \pm \xi^2 \pm \xi X_\kappa} > 0 \\ \xi \equiv \sqrt{1 - 2m/R} \\ X_k \equiv \sqrt{1 - 3m/R \pm 2a\omega_s} \end{cases} \tag{68}$$

The rotated angle after a revolution (with its corresponding sign for the respective *direct* or *retrograde* orbit) turns out to be

$$\Delta\alpha = \pm 2\pi \frac{\omega_s}{\xi^3} \left[-a(a\omega_s \pm \xi^2) \pm \frac{R^2 \omega_s \xi^2 (\xi^2 + a^2/R^2)}{a\omega_s \pm \xi^2 \pm \xi X_k} \right] \tag{69}$$

It can be easily cheked that the reduction $a = 0$ provides the results obtained for Schwarzschild case. As we already did for Schwarzschild case, the evaluation of the paralell transport for the connectors in the quotient metric of Kerr space-time lead to an equation like (61) with

$$P^i = \pm \frac{\omega_s(\xi^4 - \omega_s^2 a^2)}{X_k \xi^3 R} (0, 1, 0) \tag{70}$$

And the rotated angle by the connectors is (with the good Minkowkian limit)

$$\Delta\beta = \pm 2\pi \left[1 - \frac{(\xi^4 - \omega_s^2 a^2)}{\xi^3} \right] \tag{71}$$

Since it is fulfilled that $\omega_s C - (\xi^4 - \omega_s^2 a^2) = -X_k \xi^3$, then we obtain that the total precession angle is

$$\Delta\alpha + \Delta\beta = \mp 2\pi \left[\sqrt{1 - 3m/R \pm 2a\omega_s} - 1 \right] \tag{72}$$

APPENDIX

In this Appendix we want to show the fundamental aspects of a differential precession equation on a three-dimensional Riemannian manifold $(\mathcal{V}_3, g_{ij})$, that is to say, a differential equation of the following type):

$$\frac{dN^i}{d\tau} = \mathcal{A}^i_k(\tau)\, N^k \quad , \quad \mathcal{A}_{jk} \equiv g_{ij}\mathcal{A}^i_k = -\mathcal{A}_{kj} \tag{A1}$$

The general solution of this kind of equation is

$$N^i(\tau) = \Phi^i_k(\tau)\, N^k_{(0)} \tag{A2}$$

where

$$\begin{cases} \dot{\Phi}^i_j = \mathcal{A}^i_k(\tau)\, \Phi^k_j \\ \Phi^i_j(0) = \delta^i_j \end{cases} \tag{A3}$$

Although it is quite well known we show that (A1) is an authentic equation of precession. Indeed, by introducing the dual vector (up to a sign) of $\mathcal{A}_{jk}$ in the sense of g_{ij}

$$\Omega^i \equiv -\frac{1}{2}\,\eta^{ijk}\,\mathcal{A}_{jk} \quad , \quad \eta^{ijk} = \frac{1}{\sqrt{g}}\epsilon^{ijk} \tag{A4}$$

we see that

$$\mathcal{A}^i_k N^k = \eta^{ijk}\,\Omega_j N_k \equiv +\,(\vec{\Omega}\wedge\vec{N}\,)^i \tag{A5}$$

where the vectorial product is also understood in the sense of g_{ij}.

On the other hand the matrix $\mathcal{A}^i_k$ have the following and well-known interesting property (because of the antisymmetric character of $\mathcal{A}_{jk}$)

$$\mathcal{A}^i_j\,\mathcal{A}^j_k\,\mathcal{A}^k_l = -\Omega^2\,\mathcal{A}^i_l \quad , \quad (\mathcal{A}^3 = -\Omega^2\,\mathcal{A}) \tag{A6}$$

where

$$\Omega^2 \equiv \Omega^i\,\Omega_i = -\frac{1}{2}\mathcal{A}^i_j\,\mathcal{A}^j_i \equiv -\frac{1}{2}\mathrm{tr}\,\mathcal{A}^2 \tag{A7}$$

Let us now suppose that all components of $\mathcal{A}_{ij}$ are *constants.* Then, the general solution for (A3) is

$$\Phi(\tau) = e^{\mathcal{A}\,\tau} \tag{A8}$$

By virtue of (A6) we have

$$\Phi(\tau) = I + \frac{\sin\Omega\tau}{\Omega}\,\mathcal{A} + \frac{1-\cos\Omega\tau}{\Omega^2}\,\mathcal{A}^2 \tag{A9}$$

and therefore, the solution of (A1) can be written as follows

$$\vec{N}(\tau) = \vec{N}_0 + \frac{\sin\Omega\tau}{\Omega}\,\vec{\Omega}\wedge\vec{N}_0 \\ + \frac{1-\cos\Omega\tau}{\Omega^2}\,\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{N}_0) \tag{A10}$$

which can be simplified to give

$$\vec{N}(\tau) = (\vec{n}\cdot\vec{N}_0)\,\vec{n} + \sin\Omega\tau\,(\vec{n}\wedge\vec{N}_0) \\ - \cos\Omega\tau\,\left[(\vec{n}\cdot\vec{N}_0)\,\vec{n} - \vec{N}_0\right] \tag{A11}$$

being

$$\vec{n} \equiv \frac{\vec{\Omega}}{\Omega} \quad , \quad \vec{n}\cdot\vec{N}_0 \equiv g_{ik}n^i N_0^k \tag{A12}$$

This exppression shows that the angle between $\vec{\Omega}$ and $\vec{N}$ remains unchanged

$$\vec{n}\cdot\vec{N}(\tau) = \vec{n}\cdot\vec{N}_0 \quad , \quad \forall\tau \tag{A13}$$

So, if we take for instance $\vec{n}\cdot\vec{N}_0 = 0$, that leads to the final expression

$$\vec{N}(\tau) = N_0\,(\cos\Omega\tau\;\vec{n}_1 + \sin\Omega\tau\;\vec{n}_2) \tag{A14}$$

being $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ a triad of orthonormal vectors with the following notation

$$\begin{cases} \vec{n}_1 \equiv \dfrac{\vec{N}_0}{N_0} \\ \vec{n}_2 \equiv \vec{n}_3 \wedge \vec{n}_1 \\ \vec{n}_3 \equiv \vec{n} \end{cases} \tag{A15}$$

Finally, it is clear from above (A14) that the *precession angle* after a value of the parameter τ is equal to

$$\Delta\alpha = \Omega\,\tau \tag{A16}$$

which is a positive defined quantity because Ω is the norm of the dual vector Ω^i. Nevertheless, it is usefull in practice to assign a sign (positive or negative) to this angle, as it is done in this article, according to the rotation defined by the vector $\vec{\Omega}$ (from $\vec{n}_1$ to $\vec{n}_2$) be *direct* or *retrograde* for each particular context.

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